A MULTISTRUCTURE CREATED BY COVERINGS OF A SET

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Abstract: This paper focuses on a certain construction of a multistructure on the set of all coverings on the universal set $\mathcal{U}$ and discusses properties of this multistructure. The construction itself uses a concept dual to the *Ends Lemma,* called *Beginnings Lemma,* which is proved in the paper.

Keywords: Coverings, refinement, quasi-ordered semigroup, (quasi-)hypergroup, Ends Lemma, Beginnings Lemma.

1 INTRODUCTION

The concept of covering of spaces (on an abstract set) plays an important role in general topology. Using this notion various topological concepts and classes of significant spaces such as compact spaces, Lindelöf spaces, paracompact spaces, Čech-complete spaces, uniform spaces are defined and investigated. Coverings and their properties are also useful in some branches of algebraic topology - formerly called combinatorial topology. Moreover, by using coverings one can generalize and study rough sets theory established in the 1982 by professor Z. Pawlak, [8].

In this paper we examine properties of a certain hyperoperation $*$ on the power set of an universal set $\mathcal{U}$. There is presented *Beginnings Lemma* concept, which is dual to the concept of the *Ends Lemma.* Further, the *Beginnings Lemma,* together with refinement of a covering and principal beginning on an quasi-ordered set, plays an important role in verifying associativity property of the hyperopration $*$.

2 USED CONCEPTS

For understanding the following text it is crucial to recall some basic definitions and theorems used later in the this paper.

Definition. A *covering* of the universal set $\mathcal{U}$ is a non-empty system of sets $\mathscr{C} \subset P^*(\mathcal{U})(=P(\mathcal{U})\setminus\{\emptyset\})$ with the property $\mathcal{U} = \bigcup \mathscr{C}$.

The original Pawlak's rough set theory uses equivalence relation which induces partition into blocks of equivalence. By using coverings we generalize this concept. In fact, if for all $C_1 \neq C_2$, where C_1, C_2 are elements in a covering $\mathscr{C}$, we have $C_1 \cap C_2 = \emptyset$, then $\mathscr{C}$ is a set of blocks of equivalence.

Definition. A covering $\mathscr{S}$ is called a *refinement* of a covering $\mathscr{C}$ if each $S \in \mathscr{S}$ is a subset of at least one $C \in \mathscr{C}$. We write $\mathscr{S} \rho \mathscr{C}$, c.f. [5].

Sometimes, e.g. in [5], the symbol $<$ is used for refinement instead of ρ. Although it is easy to see that the refinement is a quasi-ordering, proofs of reflexivity and transitivity of the refinement are included later in this contribution.

Definition. A triad $(S, \cdot, \leq)$ is called *quasi-ordered semigroup* if $(S, \cdot)$ is a semigroup, $(S, \leq)$ is a quasi-ordered set and $x \leq y$ implies $x \cdot z \leq y \cdot z$ and $z \cdot x \leq z \cdot y$ for all $x, y, z \in S$. Moreover, if relation $\leq$ is antisymmetrical, then $\leq$ is ordering.

Remark: We denote reflexive and transitive relation as quasi-ordering although in literature it is denoted as preorder.

Example: The ordered triad $(\mathbb{N},\cdot,\leq)$ *is an ordered semigroup. Here the symbol "$\cdot$" stands for usual multiplication.*

Definition. The set H equipped with a hyperoperation $*$

$$*: H\times H \to P^*(H)$$

is called a *hypergroupoid*. In addition, if the hyperoperation $*$ satisfies the reproduction axiom

$$x*H = H = H*x$$

for all $x\in H$, then the structure $(H,*)$ is called a *quasi-hypergroup* (as it is defined in [3]), and if moreover a quasi-hypergroup is associative, then $(H,*)$ is called *hypergroup* (in the sense of Marty), [1]. If the hyperoperation $*$ is commutative, then $(H,*)$ is called *commutative hypergroup*. Moreover, if the hyperoperation $*$ satisfy condition $x,y\in x*y$ for all $x,y\in H$, then $(H,*)$ is called *commutative extensive hypergroup.*

Example: The set $\{1,2,\dots,n\}$, *for an arbitrary* $n\in\mathbb{N}$, *with usual ordering* $\leq$ *and a hyperoperation* $\circ:\{1,2,\dots,n\}\times\{1,2,\dots,n\}\to P^*(\{1,2,\dots,n\})$ *defined by*

$$a\circ b = \{x\in\{1,2,\dots,n\},\ \min\{a,b\}\leq x\leq \max\{a,b\}\}$$

for all $a,b\in\{1,2,\dots,n\}$ *is a commutative, extensive hypergroup. It is easy to see, that commutativity and extensivity of the hyperoperation are fulfilled. Further, for any element* $x\in\{1,2,\dots,n\}$ *such that* $1<x<n$ *we have* $1\circ x=\{1,\dots,x\}$ *and* $x\circ n=\{x,\dots,n\}$. *Therefore* $x\circ\{1,2,\dots,n\} = \{1,\dots,x\}\cup\{x,\dots,n\} = \{1,2,\dots,n\}$. *If* $x=1$ *(or* $x=n$*), then we have* $x\circ n=\{1,2,\dots,n\}$ *(or* $1\circ x=\{1,2,\dots,n\}$*). Hence reproduction axiom is satisfied. Because a verification of the associativity condition is too much time consuming we present only one of the six possible ordering of the elements* $a,b,c\in\{1,2,\dots,n\}$, *i.e.* $a\leq b\leq c$. *The verification of the associativity condition for this triad is*

$$(a\circ b)\circ c = \{a,\dots,b\}\circ c = \{a,\dots,c\} = a\circ\{b,\dots,c\} = a\circ(b\circ c).$$

Hence, for elements $a\leq b\leq c$ *associativity condition holds.*

Definition. Let $(S,\leq)$ be a quasi-ordered set, then $[a)_\leq = \{s\in S; a\leq s\}$ is called *principal end* (generated by $a\in S$), and $(a]_\leq = \{s\in S; s\leq a\}$ is called *principal beginning* (generated by $a\in S$).

There are many different concepts of constructions of semihypergroups or hypergroups such as by using (quasi-)ordered semigroup or *Ends Lemma,* [2, 6, 7].

Theorem. If $(S,\cdot,\leq)$ is quasi-ordered semigroup, then the binary hyperoperation $*: S\times S\to P^*(S)$ defined by $a*b=[a\cdot b)_\leq$ is associative for all $a,b\in S$.
For proof see [2]., or see the dual proof below.

It is convenient for us to use the dual construction to the Theorem above, which we will call *Beginnings Lemma.*

Theorem. If $(S,\cdot,\leq)$ is quasi-ordered semigroup, then the binary hyperoperation $*: S\times S\to P^*(S)$ defined by $a*b=(a\cdot b]_\leq$ is associative for all $a,b\in S$.

Proof. Let $a,b,c\in S$ be arbitrary elements. Now we show that the equation

$$\bigcup_{t\in(b\cdot c]_\leq}(a\cdot t]_\leq = \bigcup_{x\in(a\cdot b]_\leq}(x\cdot c]_\leq \tag{1}$$

holds. Let $s \in \bigcup_{t \in (b\cdot c]_\leq} (a \cdot t]_\leq$, i.e. $s \leq a \cdot t_0$ for a suitable element $t_0 \in S, t_0 \leq b \cdot c$.

Then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ and $a \cdot t_0 \leq a \cdot (b \cdot c)$. If we set $x_0 = a \cdot b$, then $s \leq x_0 \cdot c, x_0 \in (a \cdot b]_\leq$ since $s \leq a \cdot t_0 \leq a \cdot b \cdot c$. Thus $s \in (x_0 \cdot c]_\leq \subseteq \bigcup_{x \in (a \cdot b]_\leq} (x \cdot c]_\leq$.

The proof of the converse inclusion is analogous to the one above. As a result the equality (1) holds. Now with respect to (1) we obtain

$$a * (b * c) = \bigcup_{t \in b*c} a * t = \bigcup_{t \in (b \cdot c]_\leq} (a \cdot t]_\leq = \bigcup_{x \in (a \cdot b]_\leq} (x \cdot c]_\leq = \bigcup_{x \in a*b} x * c = (a*b)*c, \tag{2}$$

hence the hyperoperation $*$ is associative. □

3 APPLICATION OF THE BEGINNINGS LEMMA

Let $\mathcal{U}$ be a non-empty set and denote by $\mathrm{Cov}(\mathcal{U})$ the set of all coverings of $\mathcal{U}$.

Let us consider a mapping $\sqcup : \mathrm{Cov}(\mathcal{U}) \times \mathrm{Cov}(\mathcal{U}) \to \mathrm{Cov}(\mathcal{U})$ defined, for all $\mathscr{S}, \mathscr{C} \in \mathrm{Cov}(\mathcal{U})$,

$$\mathscr{S} \sqcup \mathscr{C} = \{S \cup C; S \in \mathscr{S}, C \in \mathscr{C}\}. \tag{3}$$

Moreover, define a hyperoperation $* : \mathrm{Cov}(\mathcal{U}) \times \mathrm{Cov}(\mathcal{U}) \to P^*(\mathrm{Cov}(\mathcal{U}))$ by

$$\mathscr{S} * \mathscr{C} = \{\mathscr{T} \in \mathrm{Cov}(\mathcal{U}), \forall T \in \mathscr{T}; \exists P \in \mathscr{S} \sqcup \mathscr{C}; T \subseteq P\} \tag{4}$$

for all $\mathscr{S}, \mathscr{C} \in \mathrm{Cov}(\mathcal{U})$.

Proposition: The hypergroupoid $(\mathrm{Cov}(\mathcal{U}), *)$ is a commutative, extensive quasi-hypergroup.

Proof. **Commutativity** of the operation of union of sets in (3) implies that operation $\sqcup$ is also commutative. Moreover, for all $P \in \mathscr{S} \sqcup \mathscr{C}$ also $P \in \mathscr{C} \sqcup \mathscr{S}$ and it follows that hyperoperation $*$ is commutative on the set $\mathrm{Cov}(\mathcal{U})$. Therefore, $(\mathrm{Cov}(\mathcal{U}), *)$ is commutative.

Reproduction Axiom. Commutativity of (4) implies that if $a * H = H$ holds, then the condition $H * a = H$ is also satisfied. Further, since $\mathrm{Cov}(\mathcal{U})$ is a set of all coverings of $\mathcal{U}$, then $\{\mathcal{U}\} \in \mathrm{Cov}(\mathcal{U})$. Moreover, for an arbitrary element $\mathscr{S} \in \mathrm{Cov}(\mathcal{U})$, we have that $S \cup \mathcal{U} = \mathcal{U}$ for all $S \in \mathscr{S}$. From here we have that $\mathscr{S} \sqcup \{\mathcal{U}\} = \{\mathcal{U}\}$. Now, since $\mathscr{S} \sqcup \{\mathcal{U}\} = \{\mathcal{U}\}$, then

$$\mathscr{S} * \{\mathcal{U}\} = \{\mathscr{T} \in \mathrm{Cov}(\mathcal{U}); \forall T \in \mathscr{T} \exists P \in \{\mathcal{U}\}; T \subseteq P\} = \{\mathscr{T} \in \mathrm{Cov}(\mathcal{U}); T \in \mathscr{T}; T \subseteq \mathcal{U}\} = \mathrm{Cov}(\mathcal{U}).$$

Therefore, $\mathscr{S} * \mathrm{Cov}(\mathcal{U}) = \mathrm{Cov}(\mathcal{U})$ for all $\mathscr{S} \in \mathrm{Cov}(\mathcal{U})$, i.e. $(\mathrm{Cov}(\mathcal{U}), *)$ is a quasi-hypergroup.

Extensivity. Let us consider two coverings $\mathscr{S}, \mathscr{C} \in \mathrm{Cov}(U)$ in the form $\mathscr{S} = \{S_i, j \in I\}$ and $\mathscr{C} = \{C_j, j \in J\}$, where I, J are non-empty sets. Let S be an arbitrary element of the $\mathscr{S}$ and denote by $M = S \cup C_{j_0}$, where $C_{j_0} \in \mathscr{C}$ is an arbitrary set. Then $S \subseteq S \cup C_{j_0} = M \in \mathscr{S} \sqcup \mathscr{C}$.

Thus $(\mathrm{Cov}(\mathcal{U}), *)$ is a commutative extensive quasi-hypergroup. □

For the completenes of considerations we give the proof of the following proposition which is in fact contained in [4, 5]

Proposition. The binary relation $\rho \subset \mathrm{Cov}(\mathcal{U}) \times \mathrm{Cov}(\mathcal{U})$ (refinement) is a quasi-ordering.

Proof. The relation ρ is reflexive: for all $S \in \mathscr{S}$ we have $S \subseteq S$, and transitive: if $\mathscr{S} \rho \mathscr{C}$ and $\mathscr{C} \rho \mathscr{T}$, then for any $S \in \mathscr{S}$ there exists $C \in \mathscr{C}$ such that $S \subseteq C$ and for C there exists $T \in \mathscr{T}$ such that $C \subseteq T$, thus $S \subseteq T$, hence $\mathscr{S} \rho \mathscr{T}$. □

Proposition. The ordered triad $(\mathrm{Cov}(\mathcal{U}), \sqcup, \rho)$ is a quasi-ordered semigroup.

Proof. We show that the inclusion $(\mathscr{S} \sqcup \mathscr{C}) \sqcup \mathscr{T} \subseteq \mathscr{S} \sqcup (\mathscr{C} \sqcup \mathscr{T})$ holds for all coverings $\mathscr{S}, \mathscr{C}, \mathscr{T} \in \mathrm{Cov}(\mathcal{U})$. Suppose that $X \in (\mathscr{S} \sqcup \mathscr{C}) \sqcup \mathscr{T}$. Then there exist elements $P \in (\mathscr{S} \sqcup \mathscr{C}), T \in \mathscr{T}$ such that $X = P \cup T$. Moreover, if $P \in \mathscr{S} \sqcup \mathscr{C}$ then there exist $S \in \mathscr{S}, C \in \mathscr{C}$ with the property $S \cup C = P$ and we can write $X = P \cup T = S \cup C \cup T$. Further, if we denote by Q the union of elements $C \cup T$, then we can write $X = S \cup Q$, i.e. there exist $S \in \mathscr{S}$ and $Q \in \mathscr{C} \sqcup \mathscr{T}$ such that $X = S \cup Q$. Hence $X \in \mathscr{S} \sqcup (\mathscr{C} \sqcup \mathscr{T})$ and the inclusion $(\mathscr{S} \sqcup \mathscr{C}) \sqcup \mathscr{T} \subseteq \mathscr{S} \sqcup (\mathscr{C} \sqcup \mathscr{T})$ is proved.
Now suppose that $X \in \mathscr{S} \sqcup (\mathscr{C} \sqcup \mathscr{T})$. Then there exists an element $Q \in (\mathscr{C} \sqcup \mathscr{T})$ and an element $S \in \mathscr{S}$ such that $X = S \cup Q$. Further, if $Q \in (\mathscr{C} \sqcup \mathscr{T})$, then there exist $C \in \mathscr{C}, T \in \mathscr{T}$ with the property $C \cup T = Q$ and we can write $X = S \cup Q = S \cup C \cup T$. Moreover, by denoting $P = S \cup C$ we can write $X = P \cup T$, i.e. there exist $P \in (\mathscr{S} \sqcup \mathscr{C})$ and $T \in \mathscr{T}$ such that $X = P \cup T$. Hence $X \in (\mathscr{S} \sqcup \mathscr{C}) \sqcup \mathscr{T}$ and the inclusion $(\mathscr{S} \sqcup \mathscr{C}) \sqcup \mathscr{T} \supseteq \mathscr{S} \sqcup (\mathscr{C} \sqcup \mathscr{T})$ holds, i.e. $(\mathscr{S} \sqcup \mathscr{C}) \sqcup \mathscr{T} = \mathscr{S} \sqcup (\mathscr{C} \sqcup \mathscr{T})$.

Let us suppose covering $\mathscr{S}_1, \mathscr{S}_2, \mathscr{C}_1, \mathscr{C}_2 \in \mathrm{Cov}(\mathcal{U})$ such that $\mathscr{S}_1 \rho \mathscr{C}_1$, $\mathscr{S}_2 \rho \mathscr{C}_2$. Then for all $S_1 \in \mathscr{S}_1$ there exists $C_1 \in \mathscr{C}_1$ with $S_1 \subseteq C_1$ and for all $S_2 \in \mathscr{S}_2$ there exists $C_2 \in \mathscr{C}_2 : S_2 \subseteq C_2$. Then if we suppose that $S \in \mathscr{S}_1 \sqcup \mathscr{S}_2$ is an arbitrary set, i.e. $S = S_1 \cup S_2$ for suitable sets $S_1 \in \mathscr{S}_1, S_2 \in \mathscr{S}_2$ then there are $C_1 \in \mathscr{C}_1, C_2 \in \mathscr{C}_2$ such that $S_i \subseteq C_i, i = 1, 2$, hence $S = S_1 \cup S_2 \subseteq C_1 \cup C_2 \in \mathscr{C}_1 \sqcup \mathscr{C}_2$, which means that $(\mathscr{S}_1 \sqcup \mathscr{S}_2) \rho (\mathscr{C}_1 \sqcup \mathscr{C}_2)$.

Hence $(\mathrm{Cov}(\mathcal{U}), \sqcup, \rho)$ is a quasi-ordered semigroup. □

The hyperoperation $*$ in (4) can be rewritten using quasi-ordering ρ in the following way

$$\mathscr{S} * \mathscr{C} = \{\mathscr{T} \in \mathrm{Cov}(\mathcal{U}); \mathscr{T} \rho (\mathscr{S} \sqcup \mathscr{C})\} = (\mathscr{S} \sqcup \mathscr{C}]_\rho. \tag{5}$$

Proposition. The commutative, extensive quasi-hypergroup $(\mathrm{Cov}(\mathcal{U}), *)$ is associative, i.e. it is a commutative hypergroup.

Proof. An ordered triad $(\mathrm{Cov}(\mathcal{U}), \sqcup, \rho)$ is quasi-ordered semigroup. Since $\mathscr{S} * \mathscr{C} = (\mathscr{S} \sqcup \mathscr{C}]_\rho$ for any pair $\mathscr{S}, \mathscr{C} \in \mathrm{Cov}(\mathcal{U})$, using the *Beginnings Lemma* we obtain that the hyperoperation $*$ is associative.

Hence the hypergroupoid $(\mathrm{Cov}(\mathcal{U}), *)$ is associative, so it is a hypergroup. □

A better insight into calculation with covering of sets yield concrete examples of a combinatorial character presented in the form of the following lemma. Let us note that covering approximation space plays an important role in generalization of classic Pawlak's rough set theory.

Lemma. For $\mathscr{S}, \mathscr{C} \in \mathrm{Cov}(\mathcal{U})$, where $\mathscr{S} \neq \mathscr{C}$, we have
(i) for $|\mathcal{U}| \in \{2, 3\}$, there is $\mathscr{S} * \mathscr{C} = \mathrm{Cov}(\mathcal{U})$;
(ii) and for $|\mathcal{U}| \geq 4$ there is $\mathscr{S} * \mathscr{C} \neq \mathrm{Cov}(\mathcal{U})$ in general.

Proof. *(i)* For $|\mathcal{U}| \in \{2, 3\}$ and any $S \in \mathscr{S}$ such that $|S| \geq 2$ there has to exist a set $C \in \mathscr{C}$ with the property $\overline{S} \subseteq C$ because $\mathscr{C}$ is a covering and has to cover all element in $\mathcal{U}$. Here $\overline{S}$ stands for complement of S. Therefore, $S \cup C = \mathcal{U}$ and $\mathcal{U} \in \mathscr{S} \sqcup \mathscr{C}$ which implies $\mathscr{S} * \mathscr{C} = \mathrm{Cov}(\mathcal{U})$. In conclusion, if any subset of $\mathscr{S}$ (or $\mathscr{C}$ respectively) has two or three elements, then $\mathscr{S} * \mathscr{C} = \mathrm{Cov}(\mathcal{U})$. The only possibility that remains is that the cardinality of all subsets S, C of coverings $\mathscr{S}, \mathscr{C}$ equal to one, i.e. $|S| = |C| = 1$. Which is in contradiction with assumption of the Lemma, that $\mathscr{S} \neq \mathscr{C}$.
Hence the part *(i)* of the above Lemma is proved.

(ii) Suppose that $\mathcal{U} = \{1, 2, 3, 4, 5 \dots\}$ and

$$\mathscr{S} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5, \dots\}\} \text{ and } \mathscr{C} = \{\{1, 2\}, \{3, 4\}, \{5, \dots\}\}.$$

Then obviously $\mathscr{S} \neq \mathscr{C}$ and $\mathscr{S}, \mathscr{C} \in \mathrm{Cov}(\mathcal{U})$. Further, by the operation $\sqcup$ we obtain

$$\mathscr{S} \sqcup \mathscr{C} = \{\{1,2\},\{1,3,4\},\{1,5,\dots\},\{2,3,4\},\{2,5,\dots\},\{1,2,3\},\{3,4\},\{3,5,\dots\},\\ \{1,2,4\},\{4,5,\dots\}\}.$$

Now, for a covering $\mathscr{T} = \{\mathcal{U}\}$, which is included in $\mathrm{Cov}(\mathcal{U})$, there does not exist any $P \in \mathscr{S} \sqcup \mathscr{C}$ such that $\mathcal{U} \subseteq P$. Hence $\mathscr{T} \notin \mathscr{S} \sqcup \mathscr{C}$, which immediately implies that $\mathscr{S} * \mathscr{C} \neq \mathrm{Cov}(\mathcal{U})$. □

4 CONCLUSION

The Ends Lemma has been used many times and also generalized in a series of papers mainly by Michal Novák - later with his collaborator Štěpán Křehlík. The contents of this contribution is based on the dual approach.

Applications of the Beginnings Lemma simplifies the proof of associativity of the defined hyperoperation. The direct calculation with coverings and corresponding refinements seems to be complicated. The direct proof of the Beginning Lemma is in fact dual to the proof of the Ends Lemma which is more elegant.

Another crucial term of our paper is the concept of the refining of coverings, which plays an important role in the general topology. The presented approach enables some applications in the rough sets theory, in particular concerning certain generalizations of Pawlak's approximations and it is relating to considerations containing extensions of the classic rough set model.

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